A Gentle Introduction to Exponential Sums in Function Fields

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Modular Hyperbolas

Let's start by taking a look at the curve

$$xy \equiv 1 \pmod{q}, \quad 0 < x, y < q$$

for some integer q. We could refer to this as a "modular hyperbola".

We can make some basic observations:

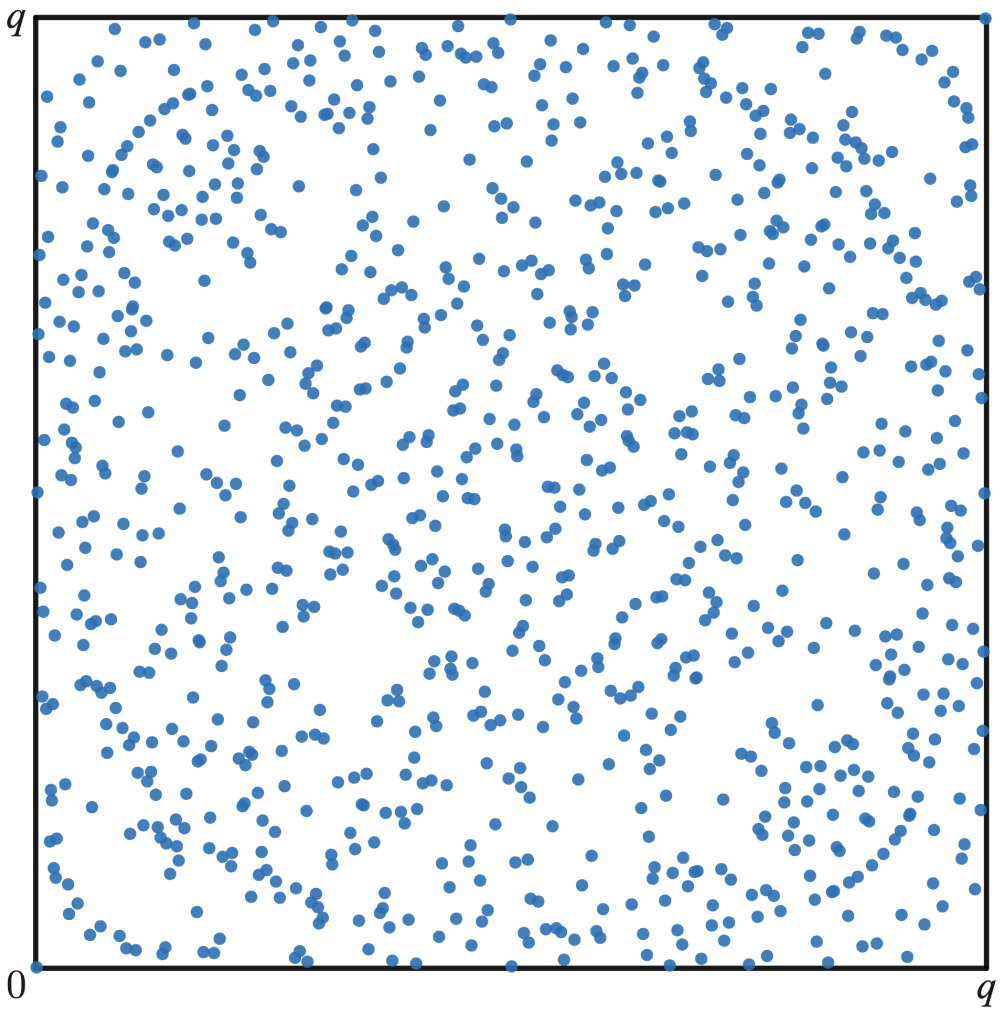
- Every value of x coprime to q has a unique inverse mod q, so the number of points on the hyperbola will be $\phi(q)$.
- There are a few lines of symmetry, such as

$$(x, y) \to (-x, -y)$$

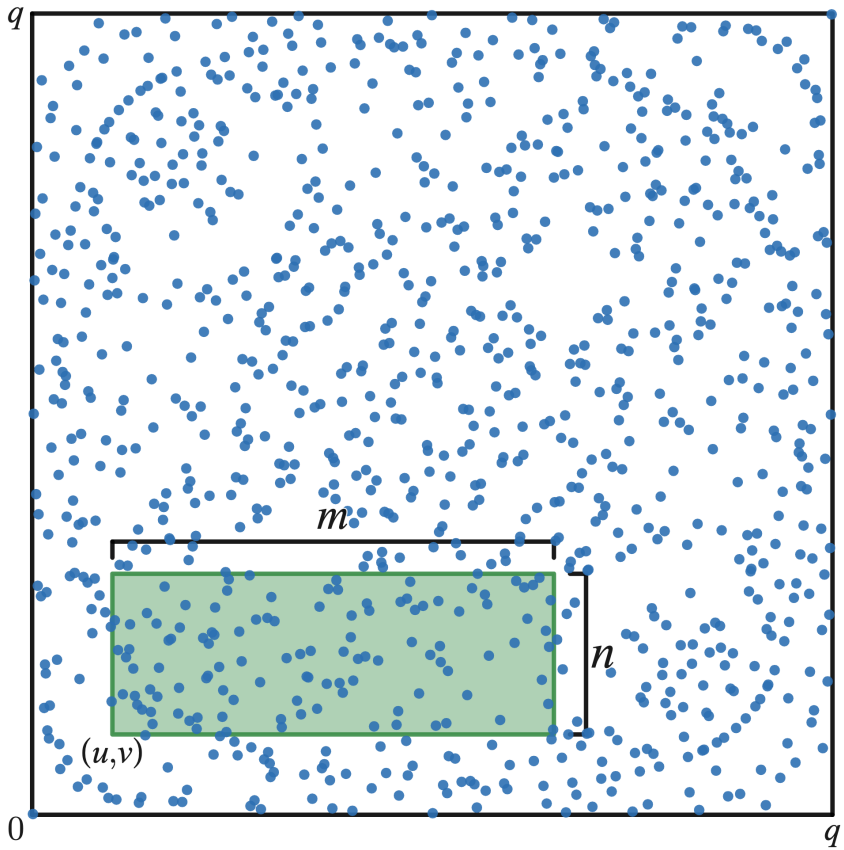
$$(x, y) \to (y, x)$$

Modular Hyperbolas

$$q = 997$$



Distribution of points



Suppose we were interested in counting points in a box.

As q grows, maybe we can get a handle on this regardless of the position of the box.

We might expect it to get its "fair share".

$$\text{Expectation: total points} \times \frac{\text{area in green}}{\text{total area}} = \phi(q)\frac{mn}{q^2}$$

Hyperbolas

Let $H_q(m, n)$ count the number of points in the box.

Theorem

$$H_q(m, n) = \phi(q)\frac{mn}{q^2} + O(q^{1/2+o(1)}).$$

(see Igor's survey article "Modular Hyperbolas" for references).

This gives us our expectation when $mn > q^{3/2}$.

The idea behind the proof

How do you prove this? Well, it turns out...

$$H_q(m,n) = \frac{1}{q^2}\sum_{t=0}^{q-1}\sum_{s=0}^{q-1}\sideset{}{^\star}\sum_{x=1}^{q-1} e\left(\frac{tx+sx^{-1}}{q}\right)\sum_{w=u}^{u+m-1} e\left(\frac{-tw}{q}\right)\sum_{z=v}^{v+n-1} e\left(\frac{-sz}{q}\right)$$

where $e(x) = e^{2i\pi x}$.

- Kloosterman sums $\sideset{}{^\star}\sum_{x=1}^{q-1} e\left(\frac{tx+sx^{-1}}{q}\right) \ll q^{1/2+o(1)}\gcd(s,t,q)^{1/2}$.
- Linear Weyl sums $\sum_{w=u}^{u+m-1} e\left(\frac{-tw}{q}\right) \ll \min\left\{m, \frac{1}{2\|t/q\|}\right\}$.
- All together, a bilinear form with Kloosterman sums.

Important: $e(x/q)$ is a character of $\mathbb{Z}/q\mathbb{Z}$, but e also allows us to keep track of information only available when viewed over $\mathbb{Z}$ and $\mathbb{R}$ (gcd and $\|\ \|$).

Characters of $\mathbb{R}$

Let G be some (additive) group. A character of G is a homomorphism

$$\psi : G \to \{z \in \mathbb{C} : |z| = 1\}.$$

That is,

$$\psi(x + y) = \psi(x)\psi(y).$$

As seen previously, often we can represent counting problems as a "character sum"; a sum of the form

$$\sum_{x \in S} \psi(x)$$

for some $S \subseteq G$.

Since $\psi(x)$ is always on the unit circle, we have a "trivial bound"

$$\left| \sum_{x \in S} \psi(x) \right| \leq |S|$$

and we are often interested in improving upon this (ideally, in the exponent).

Characters of $\mathbb{R}$

Recall that in $\mathbb{R}$, there is a "canonical" character often used:

$$e(x) := e^{2i\pi x}.$$

Why would we care about this one?

- $e(x)$ is continuous.
- All continuous (additive) characters on $\mathbb{R}$ are of the form

$$e(ax)$$

  for some $a \in \mathbb{R}$.

Why the 2π?

Building additive characters

Two keys for number theorists are the following, which rely on this scaling factor of 2π:

- It satisfies a sort of "Fourier orthogonality"

$$\int_{[0,1]} e(ax)dx = \begin{cases} 0, & a \in \mathbb{Z} \setminus 0 \\ 1, & a = 0. \end{cases}$$

- It also provides additive characters mod q. For any integer q, the function

$$x \to e\left(\frac{x}{q}\right)$$

is a non-trivial additive character of $\mathbb{Z}/q\mathbb{Z}$. This provides further orthogonality relations

$$\frac{1}{q}\sum_{x=0}^{q-1} e\left(\frac{ax}{q}\right) = \begin{cases} 0, & a \not\equiv 0 \pmod{q} \\ 1, & a \equiv 0 \pmod{q}. \end{cases}$$

Moving to function fields

We are going to move to rational function fields over finite fields, and to be able to tackle similar problems it would be nice to have a character that has similar properties:

- Continuous/ mixes well with harmonic analysis.
- Satisfies a "Fourier orthogonality".
- Can be defined "globally", but which restricts to provide "local" orthogonality.

Moving to Function Fields

Let q be a prime power, and $\mathbb{F}_q[T]$ the set of univariate polynomials over $\mathbb{F}_q$; so elements look like

$$a_n T^n + \ldots + a_1 T + a_0, \quad a_i \in \mathbb{F}_q.$$

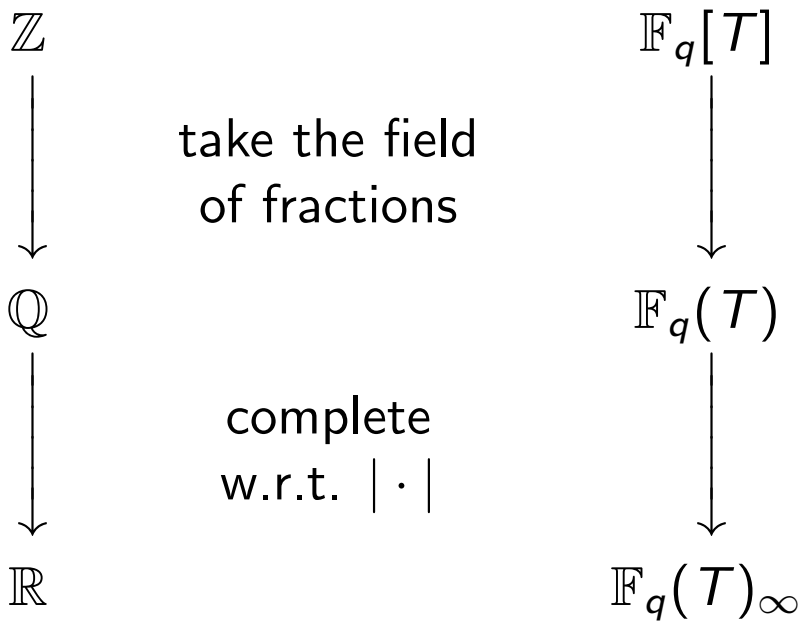
We are going to let $\mathbb{F}_q(T)_\infty$ denote the set of Laurent series in $1/T$, so elements look like

$$\sum_{-\infty}^{n} a_i T^i = a_n T^n + \ldots + a_1 T + a_0 + a_{-1} T^{-1} + a_{-2} T^{-2} + \ldots$$

$\mathbb{F}_q[T]$ very naturally sits inside $\mathbb{F}_q(T)_\infty$ (polynomials are the Laurent series with no negative powers of T), and even any rational function can naturally be written as a Laurent series in this way (think long division of polynomials).

Function Field Analogy

Think about the relationship between $\mathbb{F}_q[T]$ and $\mathbb{F}_q(T)_\infty$ like the relationship between $\mathbb{Z}$ and $\mathbb{R}$:



where on the right,

$$\left|\sum_{-\infty}^{n} a_i T^i\right| = q^n.$$

Let's first gain some intuition for this space before moving on.

"Primes"

Instead of considering prime numbers, in $\mathbb{F}_q[T]$ we can look at irreducible polynomials.

- Take Euclid's proof for the infinitude of the primes, and replace the word "prime" with "irreducible polynomial" and the proof goes through.
- The prime number theorem holds in this case:

$$\pi_q(n) = \frac{x}{\log_q x} + O(x^{1/2})$$

  where $\pi_q(n)$ counts the number of irreducible, monic polynomials of degree n, and $x = q^n$.
- The above can actually be proved just using an elementary counting argument (without RH), but GRH does also hold in this setting (Deligne, 1974).

Working modulo F

If we have some polynomial $F \in \mathbb{F}_q[T]$, the natural set of representatives of $\mathbb{F}_q[T]/\langle F \rangle$ would be

$$\{X \in \mathbb{F}_q[T] : \deg X < \deg F\}.$$

Also, recall that $\mathbb{F}_q[T]/\langle F \rangle$ is a finite field of order $q^{\deg F}$ when F is irreducible.

Intervals

The analogue of an interval

$$\{x \in \mathbb{R} \ : \ |x - y| < n\}$$

would be

$$\{X \in \mathbb{F}_q(T)_\infty : |X - Y| < q^n\}.$$

We will call something like this an "interval of length q^n".

For example, the set of all monic polyonomials of degree n can be written as

$$\{X \in \mathbb{F}_q[T] : |X - T^n| < q^n\}.$$

Another important example:

$$\{X \in \mathbb{F}_q(T)_\infty : |X| < 1\} = \{ \sum_{i=-\infty}^{-1} a_i T^i : \ a_i \in \mathbb{F}_q\}.$$

Topology

Topologically, $\mathbb{F}_q(T)_\infty$ has some interesting properties:

- Intervals are open and closed.
- Two intervals have non-empty intersection if and only if one is contained within the other.
- Most importantly, $\mathbb{F}_q(T)_\infty$ is *locally compact*, which will come up later.

Building additive characters

We now might want to get back on track and define an additive character for $\mathbb{F}_q(T)_\infty$. We will define the function

$$e_q\left(\sum_{i=-\infty}^{n} a_i T^i\right) = e^{2\pi i \ \mathrm{Tr}(a_{-1})/p}$$

where $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the absolute trace (p being the characteristic of $\mathbb{F}_q$).
As an example, if q is prime then our function becomes

$$e_q\left(\sum_{i=-\infty}^{n} a_i T^i\right) = e^{2\pi i (a_{-1})/p}$$

Why would we do this?
First, one can easily verify that this function satisfies

$$e_q(X + Y) = e_q(X)e_q(Y).$$

Continuity

It's actually not difficult to show that it is continuous.
Recall to show that a function is continuous, you need to show that the preimage of every open set is open.
The image is just a collection of discrete points on the unit circle

$$e^{2i\pi a/p}, \quad a \in \mathbb{F}_p.$$

But since the image of any element is just determined by the coefficient on T^{-1}, the preimage of one of these points is

$$S = \{\alpha T^{-1} + \sum_{i \neq -1} a_i T^i \ : \ \mathrm{Tr}(\alpha) = a\}.$$

And weirdly

$$S = \bigcup_{X \in S} \{Y \in \mathbb{F}_q(T)_\infty \ : \ |X - Y| < q^{-1}\}$$

which is of course open.

Fourier Orthogonality

Up to this point, we could've used any additive function $\mathbb{F}_q(T)_\infty \to \mathbb{F}_p$ instead of Tr. But now, recall $\mathbb{F}_q(T)_\infty$ is locally compact, which means it has a "Haar measure". This is simply a measure that is translation invariant (like the Lebesgue measure).
You don't need to worry about how this is defined, you only need to know that it is a measure and if I move a set, the measure of the set doesn't change.

After normalizing suitably, it turns out that

$$\int_{|X|<1} e_q(AX)dX = \begin{cases} 0, & A \in \mathbb{F}_q[T] \setminus \{0\} \\ 1, & A = 0. \end{cases}$$

Actually, we can get slightly better and more general orthogonality as

$$\int_{|X|<q^n} e_q(AX)dX = \begin{cases} 0, & |A| \geq q^{-n} \\ q^n, & |A| < q^{-n}. \end{cases}$$

Local Orthogonality

We also have that

$$X \to e_q\left(\frac{X}{F}\right)$$

defines a non-trivial additive character modulo F, yielding the orthogonality relation

$$\frac{1}{|F|} \sum_{\substack{X \in \mathbb{F}_q[T] \\ \deg X < \deg F}} e_q\left(\frac{AX}{F}\right) = \begin{cases} 0, & A \not\equiv 0 \pmod{F} \\ 1, & A \equiv 0 \pmod{F}. \end{cases}$$

Modular Hyperbolas (a sanity check)

Let's go back to modular hyperbolas, and see whether things work how we think they should.

Let $F \in \mathbb{F}_q[T]$ and define the modular hyperbola

$$XY \equiv 1 \pmod{F}, \quad \deg X, \deg Y < \deg F.$$

There are $\phi(F)$ points on this curve, but let's look at points in a box.

Modular Hypberolas (a sanity check)

Let $H_F(m,n)$ count the number of integral solutions to

$$XY \equiv 1 \pmod{F},\ X \in I,\ Y \in J$$

where I and J are intervals of size q^n and q^m, respectively.

What would we expect $H_F(m,n)$ to be? Again, perhaps

$$\phi(F)\frac{q^n q^m}{\left(q^{\deg F}\right)^2}.$$

Modular Hyperbolas

Theorem (B., 2023)

$$H_F(m,n) = \phi(F)q^{n+m-2\deg F} + O(q^{\deg F(1/2+o(1))})$$

How do we prove this? Well again:

$$H_F(m,n) = \frac{1}{q^{2\deg F}} \sum_{\deg T<\deg F} \sum_{\deg S<\deg F} \left(\sum_{\deg X<\deg F}^{\star} e_q\left(\frac{TX+SX^{-1}}{F}\right) \sum_{W\in I} e_q\left(\frac{-TW}{F}\right) \sum_{Z\in J} e_q\left(\frac{-SZ}{F}\right) \right)$$

If we can develop similar bounds on these Kloosterman and Weyl sums as over $\mathbb{R}$, then we can finish the proof. Hopefully, these bounds will be able to use properties of $\mathbb{F}_q[T]$ and $\mathbb{F}_q(T)_\infty$.

Weyl Sums

We will take a Weyl sum in $\mathbb{F}_q(T)_\infty$ to be a sum of the form

$$S = \sum_{\deg X < n} e_q(f(X))$$

where

$$f(X) = A_k X^k + A_{k-1} X^{k-1} + \ldots + A_1 X$$

for some $A_i \in \mathbb{F}_q(T)_\infty$.

This is a situation where actually in the simplest case function fields do much better, but in general, there are obstacles.

If $k = 1$, for a linear Weyl sum we have

$$\sum_{\deg X < n} e_q(AX) = \begin{cases} q^n, & |A| < q^{-n} \\ 0, & \text{otherwise.} \end{cases}.$$

Weyl Sums

What about $k > 1$?

Theorem (Kubota, 1971 (Weyl's lemma in $\mathbb{F}_q(T)_\infty$))

Let $f(X) = A_k X^k + ... + A_1 X + A_0$ for some $A_i \in \mathbb{F}_q(T)_\infty$. If there exists coprime $A, Q \in \mathbb{F}_q[T]$ and $B \in \mathbb{F}_q(T)_\infty$ satisfying

$$A_K = \frac{A}{Q} + B, \quad |B| < \frac{1}{Q^2}$$

and $n < \deg Q \leq (k-1)n$ then

$$\left| \sum_{\deg X < n} e_q(f(X)) \right| \leq q^{n(1-\frac{1}{2^{k-1}}+o(1))} ...$$

if

$$2 \leq k < \mathrm{char}(\mathbb{F}_q).$$

Weyl Sums

For $k > 1$, over the integers a technique called "Weyl differencing" is often used to obtain non-trivial bounds. This is what Kubota did.
It uses the idea that

$$|S|^2 = \sum_{\deg X < n} \sum_{\deg Y < n} e_q(f(X) - f(Y)) = \sum_{\deg X < n} \sum_{\deg Y < n} e_q(f(X+Y) - f(X))$$

and the degree on X in $f(X+Y) - f(X)$ should be $k-1$, so we can use an inductive argument.
But... remember we are in characteristic p, so these binomial coefficients in $f(X+Y)$ mean that actually the degree on X in $f(X+Y) - f(X)$ could be much smaller than $k-1$, and this ruins the entire argument in Weyl differencing.

Weyl Sums

Conjecture (Kubota, 1971)

The condition in the previous lemma can be relaxed to $\mathrm{char}(\mathbb{F}_q) \nmid k$.

This is still very much an open problem.

As an example, Wooley and Liu needed Weyl sums to obtain an analogue of Waring's problem in function fields.

By doing some (quite substantial) maneuvers, they were able to obtain non-trivial bounds for certain "smooth" Weyl sums, under the condition $k \nmid \mathrm{char}(\mathbb{F}_q)$.

Kloosterman sums

Next, given some $F, A, B \in \mathbb{F}_q[T]$ we can consider the Kloosterman sum

$$K_F(A,B) = \sum_{\substack{\deg X < \deg F \\ (X,F)=1}} e_q\left(\frac{AX + BX^{-1}}{F}\right).$$

Even if I have lost you up to this point, you can feel comfortable diving back in because the rest of the talk will mostly be about these Kloosterman sums.

Bounding Kloosterman Sums

Theorem (Weil, 1949)

For F irreducible and $\gcd(AB, F) = 1$ *we have*

$$\sum_{\substack{\deg X < \deg F \\ (X,F)=1}} e_q\left(\frac{AX + BX^{-1}}{F}\right) \ll q^{\deg F/2}.$$

Bounding Kloosterman Sums

With a bit more work, one can prove the following for general F, A, B, using ideas from Estermann:

$$|K_F(A, B)| \leq q^{\deg F/2+\deg \gcd(A,B,F)/2+o(\deg F)}.$$

There are a wide range of problems that can now be reduced to improving upon this in certain senses (for example, on average or in shorter intervals).

Bilinear forms with Kloosterman Sums

Recall when we were counting points on the modular hyperbola, we came across the expression

$$\frac{1}{|F|^2} \sum_{\deg T < \deg F} \sum_{\deg S < \deg F} \left(\sum_{\deg X < \deg F}^{\star} e_q\left(\frac{TX + SX^{-1}}{F}\right) \right.$$

$$\left. \sum_{W \in I} e_q\left(\frac{-TW}{F}\right) \sum_{Z \in J} e_q\left(\frac{-SZ}{F}\right) \right)$$

We could call this a bilinear form with Kloosterman sums.

Bilinear forms with Kloosterman Sums

So in general, we could consider sums like

$$\mathcal{K} = \sum_{\deg S<n} \sum_{\deg T<m} \alpha_S \beta_T K_F(S,T)$$

for some complex weights α_S, β_T.

We want to see if we can do better than just applying the triangle inequality and the Weil bound which would give

$$|\mathcal{K}| \leq \sum_{\deg S<n} \sum_{\deg T<m} |\alpha_S \beta_T| \, q^{\deg F/2+\deg \gcd(S,T,F)/2+o(\deg F)}.$$

If $|\alpha_S \beta_T| \leq 1$ then some simple inequalities give

$$|\mathcal{K}| \leq q^{m+n+\deg F(1/2+o(1))}.$$

The simplest case

In this simplest case where $\alpha_S = \beta_T = 1$, one can show (relatively simply) the following.

Theorem (B., 2023)

$$\left| \sum_{\deg S < n} \sum_{\deg T < m} K_F(S, T) \right| \leq q^{m+n+o(\deg F)} + q^{\deg F + o(\deg F)}.$$

This is non-trivial when $m, n < \deg F$ and $m + n > \deg F/2$.

Proof

The proof is actually quite straightforward: the idea is simply that

$$\sum_{\deg S<n}\sum_{\deg T<m}\sum_{\substack{\deg X<\deg F\\(X,F)=1}} e_q\left(\frac{SX+TX^{-1}}{F}\right)$$

$$=\sum_{\substack{\deg X<\deg F\\(X,F)=1}}\sum_{\deg T<m} e_q\left(\frac{TX^{-1}}{F}\right)\sum_{\deg S<n} e_q\left(\frac{SX}{F}\right).$$

These inner sums are two linear Weyl sums, and recall that we have exact orthogonality relations for these in function fields.
Applying these simply reduces the problem to counting points on a modular hyperbola.

Another bilinear form

We can also consider the slighly more general case when only $\beta_T = 1$ and α_S is arbitrary, using some ideas from Kerr, Shparlinski, Wu, Xi.
We have something more general, but for simplicity here suppose $|\alpha_S| \leq 1$. Then:

Theorem (B., 2023)

$$\sum_{\deg S<n} \sum_{\deg T<m} \alpha_S K_F(S,T) \ll q^{n+m} \left(\sum_{\deg A<\deg F-m} I_A \right)^{1/2}$$

where I_A counts the number of solutions to

$$X^{-1} + Y^{-1} \equiv A \pmod{F},\ \deg X, \deg Y < \deg F - n.$$

Of course at this point, the problem is reduced to finding sufficiently non-trivial bounds on the number of solutions I_A.

Sums of Inverses

In general, let $I_{F,A}(n)$ count solutions to

$$X^{-1} + Y^{-1} \equiv A \pmod{F}, \ \deg X, \deg Y < n.$$

We can play around with this a bit: firstly clear the denominators:

$$X + Y \equiv XYA \pmod{F}.$$

Multiply by A^{-1}, and then add and subtract A^{-2} and rearrange:

$$A^{-2} \equiv XY - XA^{-1} - YA^{-1} + A^{-2} \pmod{F}.$$

Now factor:

$$A^{-2} \equiv (X - A^{-1})(Y - A^{-1}) \pmod{F}.$$

This is a modular hyperbola! So the exact same bounds from before apply. Although this required $\gcd(A, F) = 1$.

Sums of inverses

With a bit more work, one can deal with arbitrary A (again using some techniques from Kerr, Shparlinski, Wu, Xi).

Theorem (B., 2023)

Let $d = \deg \gcd(A, F)$. *Then*

$$I_{F,A}(n) \leq q^{o(\deg F)}(q^{2n-\deg F} + q^{n+d/2-\deg F/2} + q^{\deg F/2}).$$

Proof Idea

Probably a shocker to no one at this point, it turns out $I_{F,A}(n)$ is equal to

$$q^{2n-3\deg F} \sum_{\deg U<\deg F-n} \sum_{\deg V<\deg F-n} \sum_{\deg T<\deg F} K_F(U,T)K_F(V,T)e_q\left(\frac{-AT}{F}\right).$$

This time, to deal with these Kloosterman sums, we use two important facts:

- Kloosterman sums satisfy a twisted multiplicativity: if $F = GH$ where $\gcd(G,H) = 1$ then

$$K_F(S,T) = K_G(SH^{-1}, TH^{-1})K_H(SG^{-1}, TG^{-1})$$

  so we can reduce the problem to irreducible powers.
- Kloosterman sums modulo prime powers satisfy a special relation:

$$K_{P^k}(S,T) = 2\left(\frac{C}{P}\right)^k q^{k\deg P/2}\mathrm{Re}\ \epsilon_P e_q\left(\frac{2C}{P^k}\right)$$

  if there exists some C such that $C^2 \equiv ST \pmod{P^k}$, and where ϵ_P lies on the unit circle.

Incomplete Kloosterman Sums

As a final problem to consider: if the typical Kloosterman sum is

$$\sum_{\substack{\deg X < \deg F \\ (X,F)=1}} e_q\left(\frac{SX + TX^{-1}}{F}\right)$$

then we could consider cutting it short, as in

$$\sum_{\substack{\deg X < n \\ (X,F)=1}} e_q\left(\frac{SX + TX^{-1}}{F}\right)$$

for some $n < \deg F$.

Incomplete Kloosterman Sums

You can use orthogonality relations from before to show that the Weil bound holds for these as well,

$$\sum_{\substack{\deg X<n \\ (X,F)=1}} e_q\left(\frac{SX+TX^{-1}}{F}\right) \ll q^{\deg F(1/2+o(1))}$$

which is non-trivial for $n > \deg F/2$.

It is conjectured that we might have something like

$$\sum_{\substack{\deg X<n \\ (X,F)=1}} e_q\left(\frac{SX+TX^{-1}}{F}\right) \ll q^{n/2+o(\deg F))},$$

but in general, we don't know very much beyond the Weil bound.

Incomplete Kloosterman Sums

Using some ideas of Khan, we can show the following.

Theorem (B., 2023)

Fix an integer k satisfying $2 \leq k \leq \mathrm{char}(\mathbb{F}_q)$. Then for any irreducible $P \in F_q[T]$, any $T \in \mathbb{F}_q[T]$ with $(T, P) = 1$, and any positive integer n with $\deg P < n < (k-1)\deg P$ we have

$$\sum_{\substack{\deg X < n \\ (X,P)=1}} e_q\left(\frac{TX^{-1}}{P^k}\right) \ll q^{\deg P^k(1/2-\delta_k)}$$

for some $\delta_k > 0$.

What to remember? We can do better than the Weil bound, but it requires fixed k and large characteristic.

Ideas used in the proof

Again, don't worry about the details, just the ideas here.

- Use orthogonality to relate it to an average of Kloosterman sums

$$\sum_{\substack{\deg X<n \\ (X,P)=1}} e_q\left(\frac{TX^{-1}}{P^k}\right) = q^{n-\deg P^k} \sum_{\deg S<\deg P^k-n} K_{P^k}(S,T).$$

- Recall that for prime powers

$$K_{P^k}(S,T) = 2\left(\frac{Y}{P}\right)^k q^{k\deg P/2}\mathrm{Re}\ \epsilon_P e_q\left(\frac{2Y}{P^k}\right)$$

  if there exists some Y such that $Y^2 \equiv ST \pmod{P^k}$, and where ϵ_P lies on the unit circle.

Instead of summing over S, we could try to sum over the solutions Y to $Y^2 \equiv ST \pmod{P^k}$.

Quadratic Solutions

You can make some substitutions, and it turns that you can write

$$Y^2 \equiv ST \equiv W^2(1 + XP) \pmod{P^k}$$

for some W depending on S and T and X depending only on S.
Then recall Hensel's lemma shows us how to solve these! Working through it in function fields, we can write

$$Y \equiv \pm W(1 + A_1XP + ... + A_{k-1}X^{k-1}P^{k-1}) \pmod{P^k}$$

for some A_i independent of S and T.
For our purposes, those details don't really even matter, but what does matter is that we have converted the sum over Y to a new sum over these polynomials: meaning that the problem reduces to bounding

$$\sum_{\deg X < (k-1)\deg P - n} e_q\left(2W\frac{1 + A_1XP + ... + A_{k-1}X^{k-1}P^{k-1}}{P^k}\right)...$$

another Weyl sum!

Improvements

Again, the issue of characteristic comes up.

Can we

- relax the characteristic condition?
- remove the condition that k should be fixed?

Thank you!